5.1 Polynomials in one variable

Definition 5.1.1 monomial, polynomial

Every expression

$$\dots, x^6, x^5, x^4, x^3, x^2, \underbrace{x}_{=x^1}, \underbrace{1}_{=x^0}$$

is called a **monomial in the variable** x (**Monom in der Variablen** x).
Every sum of finitely many real multiples of monomials is called a **polynomial in** x (**Polynom in** x).
Etymology: (ancient) greek μόνος mónos - englisch *single*; greek πολύ polý - englisch *many*; greek όνομα ónoma - englisch *name*; monomial = single name, polynomial = many names

Example 5.1.2. Here is an example of a polynomial in x: ✎

Example 5.1.3. Expressions such as $\frac{1}{x}$ or $\frac{x}{x^2+1}$ or $\sqrt{x}$ are not polynomials.

5.1.4. More examples of polynomials:
- $r(X) = \sqrt{5}X^3 + \frac{7}{2}$ is a polynomial in (the variable) X.
- $T(a) = a^3 + a$ is a polynomial in a.
- $f(u) = 2u^3$ is a polynomial in u.

Examples 5.1.5. Polynomials and monomials in everyday life:
- The area of a square with side length a is $F(a) = a^2$.
- The volume of a cube with side length a is $V(a) = a^3$.
- The usual “grade formula” $n(p) = 1 + 5 \cdot \frac{p}{38} = \frac{5}{38}p + 1$ is a polynomial in p; here p is the number of points and 38 is the number of points required for grade 6).

Note 5.1.6 Addition and multiplication of polynomials

If $p = p(x)$ and $q = q(x)$ are two polynomials (in the same variable x), then their
- sum $p(x) + q(x)$,
- difference $p(x) - q(x)$ and
- product $p(x)q(x) = p(x) \cdot q(x)$,

defined in the obvious way, are again polynomials.
In short: The set of polynomials is **closed under** addition, subtraction and multiplication.

Example 5.1.7. Explain «structured» calculation methods for addition/subtraction (writing underneath) and multiplication (use table) on blackboard.

Exercise E1

(a) Show that

$$\left(1 + x + x^2 + x^3 + x^4\right)(1 - x) = 1 - x^5$$

(b) For every $n \in \mathbb{N}$, show that

$$\left(1 + x + x^2 + x^3 + \dots + x^{n-1} + x^n\right)(1 - x) = 1 - x^{n+1}$$

(c) Deduce the power sum formula (Potenzsummenformel) from the above identity (another name for an equality that is always true).

Motivation 5.1.8 (standard form of a polynomial)**.** The same polynomial can be written in many ways: ✎

The highlighted representation is called **standard form** of our polynomial. It is characterized by the the following two properties:

(1) The monomial is written as a finite sum of non-zero real numbers multiplied by a monomial, and every monomial appears **at most once**.
(2) The monomials involved are **ordered in descending order** according to their exponents.

The **degree** (**Grad**) of a polynomial in standard form is the exponent of the monomial on the far left. In our example: ✎

The real numbers in front of the monomials are called **coefficients** (**Koeffizienten**) of the polynomial. In our example:

- 7 is the **coefficient at** x^4.
  This coefficient at the highest power of x is also called the **leading coefficient** (**Leitkoeffizient**) of the polynomial.
  If the leading coefficient is 1, the polynomial is called **monic** (**normiert**).
- 6 is the **coefficient at** x^3.
- The **coefficient at** $x = x^1$ is -5.
- The **coefficient at** $1 = x^0$ is 10.
  This coefficient at $1 = x^0$ is the **constant coefficient** (**konstanter Koeffizient**) of the polynomial.
- The coefficients at x^2 and x^{21} and all other monomials not showing up in standard form are 0.

Definition 5.1.9 standard form of a polynomial, degree, coefficient, leading coefficient, constant coefficient

Every polynomial $p = p(x) \neq 0$ can oviously be written in such a way that the two conditions in Motivation 5.1.8 are satisfied. We then say that the polynomial is in **standard form**.

- The **degree** (**Grad**)

$$\deg(p)$$

  of a polynomial p is the highest exponent occuring in a monomial when the monomial is written in standard form.
- The **coefficient of** p **at** x^n (**Koeffizient von** p **bei** x^n) is the the factor in front of the monomial x^n when p is written in standardform. (If the monomial x^n does not appear, the coefficient at x^n is zero by definition.)
- The **leading coefficient** (**Leitkoeffizient**) of a polynomial is the coefficient at the highers power of x when the polynomial is written in standard form.
- The polynomial is called **monic** (**normiert**) if the leading coefficient is 1.
- The **constant coefficient** (**konstante Koeffizient**) of a polynomial is the coefficient at $1 = x^0$ when the polynomial is written in standard form.

The so-called **zero polynomial** (**Nullpolynom**) $p(x) = 0$ has, by definition, the standard form $p(x) = 0$; its degree $-\infty$. (It does not have a leading coefficient and it is is not monic. All its coefficients are 0.)

5.1.10. Advantage of the standard form: If two polynomials are given in standard form, you can immediately determine whether the polynomials are equal or not by **comparing their coefficients** (**Koeffizientenvergleich**): The polynomials are equal if and only if the coefficients at each monomial coincide.

Exercise E2 For the following polynomials determine

- the standard form,
- the degree,
- the leading coefficient,
- the constant coefficient,
- the coefficients at all monomials up to degree 6 (i. e. at $1, x, x^2, x^3, x^4, x^5, x^6$ resp. at $1, t, t^2, t^3, t^4, t^5, t^6$),
- whether the polynomial is monic or not.

(a) $p(x) = 4x^4 + 3x^3 + 2x^2 + x - 1 - 2x - 3x^2 - 4x^3 - 5x^4 + (x^2 - 1)(x^2 + 1)$
(b) $r(x) = (x - 1)(1 + x + x^2 + x^3 + x^4 + x^5)$
(c) $q(t) = (t + 1)^3 - t(t^2 + 3)$

Exercise E3

(a) For $p = p(x) = x^2 + 3$ and $q = q(x) = 2x^3 + 2x^2 + x - 3$, determine the degree, the leading coefficient and the constant coefficient of $p + q$ and of $p \cdot q$.
Hint: You do not need to calculate sum and product. You may use the two «structured» calculation methods explained.

(b) Let p be a polynomial of degree 3 and q a polynomial of degree 4.
- What is the degree of $p \cdot q$? What can you say about the leading coefficient of $p \cdot q$?
- What is the degree of $p + q$? What is the degree of $p - q$?

(c) Let p and q be polynomials of degree 3.
- What is the degree of $p \cdot q$?
- What is the degree of $p + q$? (watch out!)

(d) If $\deg(p)$ denotes the degree of a polynomial p: Complete the following statements so that they are valid for all polynomials p and q in the "best possible way".

$$\deg(p \cdot q) =$$
$$\deg(p + q) \leq$$

When do we have equality in the inequality?

5.2 Polynomial long division (Polynomdivision)

5.2.1. In arithmetic, long division (with remainder; Division mit Rest) is the standard algorithm for dividing a natural number a by a natural number b.

A very similar division algorithm, called **polynomial long division** works for polynomials.

Example 5.2.2. We know that $x^2 - 1 = (x - 1)(x + 1)$.
Is it true that $x^3 - 1$ is divisible by $x - 1$ oder $x + 1$ teilbar?

Polynomial long division: ✎

The division scheme shows: $\dfrac{x^3 - 1}{x - 1} =$ ✎ resp. $x^3 - 1 =$ ✎

✎

The division scheme shows: $\dfrac{x^3-1}{x+1} =$ ✎ It is better to write the result as follows: ✎

Exercise E4 Practice polynomial long division interactively on the following webpage. You are expected to type in the solution step by step.

- http://www.arndt-bruenner.de/mathe/scripts/polynomdivisionueben.htm

Maybe demonstrate the first steps; the level can be set in the upper left window (scroll down).

Maybe also useful: http://www.arndt-bruenner.de/mathe/scripts/polynomdivision.htm

Theorem 5.2.3 Polynomial long division

Let $p = p(x)$ and $q = q(x) \neq 0$ be polynomials in the same variable x. Then there are uniquely determined polynomials $s = s(x)$ and $r = r(x)$ such that

$$p(x) = s(x) \cdot q(x) + r(x) \qquad \text{and} \qquad \deg(r) < \deg(q)$$

Equivalent, in shorter notation:

$$p = s \cdot q + r \qquad \text{and} \qquad \deg(r) < \deg(q)$$

The polynomial $r(x)$ is called the **remainder of the division**.

5.2.4. The similar statement for ordinary long division of natural numbers is: Let p and $q \neq 0$ be natural numbers. Then there are unique natural numbers s and r such that

$$p = s \cdot q + r \qquad \text{and} \qquad r < q$$

Exercise E5 Determine using polynomial long division.

a) $(9x^3 + 3x^2 - 8x + 2) : (3x - 1)$

b) $(x^3 + 6x^5 - 3x^4 + 6 - 9x^2) : (1 - 4x + 3x^3)$ Hint: Write dividend and divisor in standard form (exponents descending)!

c) $\left(2r^3 + 4r^2 - \frac{1}{2}r - 1\right) : \left(\frac{1}{3}r + \frac{2}{3}\right)$

d) Treat a and b as fixed numbers: $(8ax^3 + 4bx^2 - 2a^3x + 1) : (2x - 1)$

5.3 Lösungen

Hinweise zu den Symbolen:

Diese Aufgaben könnten (mit kleinen Anpassungen) an einer Prüfung vorkommen. Für die Prüfungsvorbereitung gilt: "If you want to nail it, you'll need it".

Diese Aufgaben sind wichtig, um das Verständnis des Prüfungsstoffs zu vertiefen. Die Aufgaben sind in der Form aber eher nicht geeignet für eine Prüfung (zu grosser Umfang, nötige «Tricks», zu offene Aufgabenstellung, etc.). **Teile solcher Aufgaben können aber durchaus in einer Prüfung vorkommen!**.

Diese Aufgaben sind dazu da, über den Tellerrand hinaus zu schauen und/oder die Theorie in einen grösseren Kontext zu stellen.

Lösung zu E1 ex-potenzsummenformel-alternativbeweis

Lösung zu E2 ex-polynom-standardform-etc

(a) $p(x). = 4x^4+3x^3+2x^2+x-1-2x-3x^2-4x^3-5x^4+(x^2-1)(x^2+1) = -x^4-x^3-x^2-x-1+x^4-1 = \boxed{-x^3-x^2-x-2}$, Grad 3, Leitkoeffizient -1, also nicht normiert, konstanter Koeffizient -2, Koeffizienten sind -2 bei 1, -1 bei x, -1 bei x^2, -1 bei x^3, und 0 bei x^4, x^5 und x^6

(b) $r(x) = (x-1)(1+x+x^2+x^3+x^4+x^5) = \boxed{x^6-1}$, Grad 6, Leitkoeffizient 1, also normiert, konstanter Koeffizient -1; Koeffizienten -1 bei $1 = x^0$, 1 bei x^6 und Koeffizient 0 bei allen anderen Monomen.

(c) $q(t) = (t+1)^3 - t(t^2+3) = t^3+3t^2+3t+1-t^3-3t = \boxed{3t^2+1}$, Grad 2, Leitkoeffizient 3, also nicht normiert, konstanter Koeffizient 1, Koeffizient 3 bei t^2, Koeffizient 1 bei 1, andere Koeffizienten 0.

Lösung zu E3 ex-polynom-grad-summe-produkt

(a) Für $p = p(x) = x^2+3$ und $q = q(x) = 2x^3+2x^2+x-3$ bestimme man den Leitkoeffizienten und den konstanten Koeffizienten von $p+q$ und von $p \cdot q$.
$p+q$ hat Grad 3. Sein Leitkoeffizient ist 2, sein konstanter Koeffizient 0.
$p \cdot q$ hat Grad 5, Leitkoeffizient 2 und konstanten Koeffizienten -9.

(b) Seien p ein Polynom vom Grad 3 und q ein Polynom vom Grad 4.
- Welchen Grad hat $p \cdot q$? 7=3+4
  Was können Sie über den Leitkoeffizienten von $p \cdot q$ aussagen? Er ist das Produkt der Leitkoeffizienten von p und q.
- Welchen Grad hat $p+q$? 4 Welchen Grad hat $p-q$? 4

(c) Seien p und q Polynome vom Grad 3
- Welchen Grad hat $p \cdot q$? 3+3=6
- Welchen Grad hat $p+q$? Achtung! Grad 3, 2, 1, 0 oder $-\infty$. Grad 3 hat die Summe genau dann, wenn die Summe der beiden Leitkoeffizienten nicht Null ist. Wenn $q=-p$ gilt, gilt $p+q=0$, was Grad $-\infty$ hat.

(d) Wenn $\deg(p)$ den Grad (englisch «degree») eines Polynoms bezeichnet: Vervollständigen Sie die folgenden Aussagen so, dass sie für alle Polynome p und q in «bestmöglicher Art» gelten.

$$\deg(p \cdot q) = \deg(p) + \deg(q)$$
$$\deg(p+q) \leq \max(\deg(p), \deg(q))$$

Wann gilt Gleichheit bei der zweiten Aussage? Gleichheit gilt genau dann, wenn
- $\deg(p) \neq \deg(q)$ gilt oder wenn
- $\deg(p) = \deg(q)$ gilt, aber die Summe der beiden Leitkoeffizienten nicht Null ist (d. h. der eine Leitkoeffizient ist nicht die Gegenzahl (= das Negative) des anderen Leitkoeffizienten).

Lösung zu E4 ex-polynomdivision-online-ueben

Die Lösungen können online kontrolliert werden.

Lösung zu E5 ex-polynomdivision-schriftlich

a) $(9x^3+3x^2-8x+2) : (3x-1) = 3x^2+2x-2$ d. h. $\frac{9x^3+3x^2-8x+2}{3x-1} = 3x^2+2x-2$

b) $(x^3 + 6x^5 - 3x^4 + 6 - 9x^2) : (1 - 4x + 3x^3) = 2x^2 - x + 3$ Rest $-15x^2 + 13x + 3$
Das kann man auch so schreiben: $\frac{x^3+6x^5-3x^4+6-9x^2}{1-4x+3x^3} = 2x^2 - x + 3 + \frac{-15x^2+13x+3}{1-4x+3x^3}$

c) $\left(2r^3 + 4r^2 - \frac{1}{2}r - 1\right) : \left(\frac{1}{3}r + \frac{2}{3}\right) = 6r^2 - \frac{3}{2}$
Die Rechnung wird einfacher, wenn man den Quotienten $\frac{2r^3+4r^2-\frac{1}{2}r-1}{\frac{1}{3}r+\frac{2}{3}}$ mit 6 erweitert und dann die Polynomdivision $(12r^3 + 24r^2 - 3r - 6) : (2r + 4)$ durchführt.

d) $(8ax^3 + 4bx^2 - 2a^3x + 1) : (2x - 1) = 4ax^2 + 2(a + b)x + (a + b - a^3)$ Rest $1 + a + b - a^3$
d. h. $\frac{8ax^3+4bx^2-2a^3x+1}{2x-1} = 4ax^2 + 2(a + b)x + (a + b - a^3) + \frac{1+a+b-a^3}{2x-1}$